Geometry, stereometry and spherical geometry

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This paper contains basic definitions, facts and formulas related to the 2d and 3d geometry useful for programming competitions. Algorithms nor geomtrical data structures are not included.

1 2d geometry

A 2d *point* is defined by its coordinates (x, y).

A 2d *vector* is also defined by its coordinates $\vec{v} = (x, y)$. We will identify points and vectors, that allows us to add vector to a point without writing extra classes and methods.

Let $\vec{v_1} = (x_1, y_1)$ and $\vec{v_2} = (x_2, y_2)$. Then:

- $\vec{v_1} + \vec{v_2} = (x_1 + x_2, y_1 + y_2)$ is the sum of vectors;
- $\vec{v_1} - \vec{v_2} = (x_1 - x_2, y_1 - y_2)$ is the difference of vectors;
- $k\vec{v_1} = \vec{v_1}k = (kx_1, ky_1)$ is the vector multiplied by the real value k;
- $\vec{v_1} \cdot \vec{v_2} = (v_1, v_2) = x_1x_2 + y_1y_2$ is the *scalar* product of vectors;
- $\vec{v_1} \times \vec{v_2} = [v_1, v_2] = x_1y_2 - x_2y_1$ is the *vector* product of vectors.

(From the mathematical point of view the vector product is a bit different thing, but we'll return to this when we discuss the 3d geometry)

For vector $\vec{v} = (x, y)$ its Euclidean norm is defined as $\sqrt{x^2 + y^2}$. If $\vec{v} \neq \vec{0}$, $\vec{v}/|v|$ always has a unit norm.

Scalar product $(\vec{u}, \vec{v})$ has the following geometrical meaning: if $u \neq \vec{0}$, $(\vec{u}, \vec{v}) = \text{proj}_{\vec{u}}\, \vec{v} \times |u|$ where $\text{proj}_{\vec{u}}\, \vec{v}$ is the signed projection length of vector $\vec{v}$ on the vector $\vec{u}$. In particular, when $\vec{u}$ has unit norm, $(\vec{u}, \vec{v})$ is a length of a component of vector $\vec{v}$ that is parallel to $\vec{u}$.

The following equations are held:

$$(\vec{u}, \vec{v}) = |v||u| \cos \angle(\vec{u}, \vec{v})$$

$$[\vec{u}, \vec{v}] = |v||u| \sin \angle(\vec{u}, \vec{v})$$

Note that $\angle(\vec{u}, \vec{v})$ depends on the order of u and v, it is calculated in counter-clockwise direction from $\vec{u}$ to $\vec{v}$ modulo 2π.

The geometrical meaning of $[\vec{u}, \vec{v}]$ is the oriented area of parallelogram between vectors $\vec{u}$ and $\vec{v}$. A triangle area is 2 times smaller than parallelogram area.

The geometrical meaning of $\text{sgn}(\vec{u}, \vec{v})$: if it is zero, $\vec{u}$ and $\vec{v}$ are orthogonal, if it is positive, the angle between them is acute, otherwise it is obtuse.

The geometrical meaning of $\text{sgn}(\vec{u}, \vec{v})$ is very important: it defines the belonging of $\vec{v}$ in respect to the line defined by $\vec{v}$. If it is zero, $\vec{u} \parallel \vec{v}$, if it is positive, then $\vec{v}$ lies to the left of the line defined by $\vec{v}$, otherwise $\vec{v}$ lies to the right of the line defined by $\vec{v}$.

NB: The words "counter-clockwise" and "left/right" change their meaning to the opposite if we define vector product as $[\vec{u}, \vec{v}] = x_2y_1 - y_2x_1$ instead of the definition above. The most common way not to make any mistake is to choose the sign of the vector product in such way that $[e_x, e_y] = 1$, where $e_x = (1, 0)$ and $e_y = (0, 1)$, this yields the formula that is written above.

The line equation: $Ax+By = C$ where at least one of A and B is not zero. This is equivalent to the following: $(A, B) \times (x, y) = C$ that shows that (A, B) is a *normal* vector of the line (i.e. vector orthogonal to the line.

The *signed distance* from the point (x_0, y_0) to the line $Ax + By = C$ is $\frac{Ax_0+By_0-C}{\sqrt{A^2+B^2}}$; it is zero if (x_0, y_0) belongs to the line, it is positive if (x_0, y_0) lies in the half-plane defined by the vector (A, B), and it is negative otherwise.

The intersection of two lines $A_1x+B_1y = C_1$ and $A_2x+B_2y = C_2$ is defined by the following formulae (Cramer's rule):

$$\Delta = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

$$\Delta_x = \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}$$

- if $\Delta = \Delta_x = \Delta_y = 0$, lines coincide;
- if $\Delta = 0$ but $\Delta_x \neq 0$ or $\Delta_y \neq 0$, lines are parallel but do not coincide;
- if $\Delta \neq 0$, then the intersection point $O = (\Delta_x/\Delta, \Delta_y/\Delta)$

Note that Δ is actually $[n_1, n_2]$ where n_1 and n_2 are normal vectors of the lines. Indeed, lines are parallel iff their normal vectors are parallel.

The baricentric line division rule: if X belongs to the line AB, then $\vec{X} = \frac{\vec{AX}}{\vec{AB}}\vec{B} + \frac{\vec{XB}}{\vec{AB}}\vec{A}$. Here $\frac{\vec{u}}{\vec{v}}$ is the proportion between two vectors belonging to the same line: it is $\frac{|u|}{|v|}$ if they have the same direction and $-\frac{|u|}{|v|}$ otherwise. Note that $\frac{\vec{AX}}{\vec{AB}} + \frac{\vec{XB}}{\vec{AB}} = \frac{\vec{AX}+\vec{XB}}{\vec{AB}} = \frac{\vec{AB}}{\vec{AB}} = 1$.

Equivalent statement: if λ is a real number, then $\lambda\vec{B} + (1 - \lambda)\vec{A} = \vec{A} + \lambda(\vec{B} - \vec{A})$ is always a point of the line AB that divides segment AB in proportion of $\lambda : (1 - \lambda)$.

The baricentric line division rule is a useful thing.

Corollary 1: if line is defined by two points $\vec{A} = (x_1, y_1)$ and $\vec{B} = (x_2, y_2)$, then its intersection with a horizontal line $y = y_0$ is defined by a following formula:

$$\frac{(y_0 - y_1)\vec{B} + (y_2 - y_0)\vec{A}}{y_2 - y_1}$$

Corollary 2: the intersection of lines AB and CD is defined by a following formula:

$$\frac{[\vec{CB}, \vec{CD}]\vec{A} + [\vec{CD}, \vec{CA}]\vec{B}}{[\vec{AB}, \vec{CD}]}$$

Angle between vectors $\vec{u}$ and $\vec{v}$ is defined by the formula $\operatorname{atan2}([\vec{u}, \vec{v}], (\vec{u}, \vec{v}))$.

For a vector $\vec{v} = (x, y)$, if we rotate it by angle ϕ counter-clockwise, we get a vector $\vec{v^\phi} = (x\cos\phi - y\sin\phi, x\sin\phi + y\cos\phi)$. In particular, rotation by 90 degrees (or by $\pi/2$) is defined by a formula $(-y, x)$.

2 3d geometry

A scalar product of $\vec{v_1} = (x_1, y_1, z_1)$ and $\vec{v_2} = (x_2, y_2, z_2)$ is $x_1x_2 + y_1y_2 + z_1z_2$.

A vector product of $\vec{v_1}$ and $\vec{v_2}$ is

$$[\vec{v_1}, \vec{v_2}] = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1)$$

, note that in 3d case the vector product is a vector!

$|[\vec{v_1}, \vec{v_2}]|$ is an area of a parallelogram built by vector $\vec{v_1}$ and $\vec{v_2}$.

Vector product is always orthogonal to both $\vec{v_1}$ and $\vec{v_2}$.

A mixed product $(\vec{v_1}, \vec{v_2}, \vec{v_3})$ is $(\vec{v_1}, [\vec{v_2}, \vec{v_3}])$. Also, by definition:

$$(\vec{v_1}, \vec{v_2}, \vec{v_3}) = (x_1, y_1, z_1) \cdot \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Geometric meaning of a mixed product is a signed volume of a parallelepiped built by v_1, v_2, v_3 as a sides. A tetrahedron volume is 6 times smaller than the parallelepiped volume.

The $\operatorname{sgn}(u, v, w)$ has several geometric meanings:

- if $\vec{u}$ is a fixed vector, then $\operatorname{sgn}(\vec{u}, \vec{v}, \vec{w})$ is zero if $\vec{v}, \vec{w}$ and $\vec{u}$ lie in the same plane, is positive if $\vec{w}$ is counter-clockwise from $\vec{v}$ when watching straight at the end of $\vec{u}$ (in direction of $-\vec{u}$), and negative otherwise;
- $\operatorname{sgn}(\vec{u}, \vec{v}, \vec{w})$ depending on which half-space w belongs in respect to the plane containing $\vec{u}$ and $\vec{v}$.

Let $\operatorname{vol}(A, B, C, D) = \operatorname{vol} ABCD = (\vec{AB}, \vec{AC}, \vec{AD})$.

The easiest way to define the line in 3d space is the choose two points on it.

A plane can be defined:

- by the equation $Ax + By + Cz = D \Leftrightarrow (A, B, C) \cdot (x, y, z) = D$;
- by three points $\vec{A}$, $\vec{B}$, $\vec{C}$ belonging to it.

The intersection of a line AB and plane CDE is defined by a baricentric rule:

$$O = \frac{\operatorname{vol} CDEB \cdot \vec{A} + \operatorname{vol} ACDE \cdot \vec{B}}{\operatorname{vol} CDEB + \operatorname{vol} ACDE}$$

If the plane is defined by a normal vector $\vec{n}$ (and the equation $(\vec{n}, (x, y, z)) = D$), then the intersection of a line AB and plane is defined by a baricentric rule:

$$O = \frac{((\vec{n}, \vec{B}) - D)\vec{A} + (D - (\vec{n}, \vec{A}))\vec{B}}{(\vec{n}, \vec{AB})}$$

The distance between the line $\vec{A}+\lambda\vec{u}$ and the line $\vec{B}+\lambda\vec{v}$ is always attained on some segment CD (C belongs to the first line, D belongs to the second line), such that $(\vec{CD},\vec{u}) = (\vec{CD},\vec{v}) = \vec{0}$. Hence, let $\vec{n} = [\vec{u},\vec{v}]/|[\vec{u},\vec{v}]|$, and $\vec{CD} = x\cdot\vec{n}$. From this equation, $x = (\vec{AB},\vec{n})$, that is the formula of distance between two non-parallel lines.

3 Spherical geometry

The main idea: keep spehere radius $r = 1$ and use Euclidean coordinate system, that makes everything simpler.

Suppose $\vec{v} = (x, y, z)$, $|\vec{v}| = 1$. The spherical coordinates are: latitude ψ and longitude ϕ.

Spherical coordinates $\to$ Euclidean coordinates:

$$z = \sin\psi$$

$$x = \cos\psi\cos\phi$$

$$y = \cos\psi\sin\phi$$

Euclidean coordinates $\to$ spherical coordinates:

$$\psi = \mathrm{atan2}(z, \sqrt{x^2+y^2})$$

$$\phi = \mathrm{atan2}(y, x)$$

Distance on sphere between points with radius-vectors $\vec{u}$ and $\vec{v}$ is $d(\vec{u},\vec{v}) = \angle(\vec{u},\vec{v}) = \mathrm{atan2}(|[\vec{u},\vec{v}]|, \vec{u}\cdot\vec{v})$.

Arc intersection on sphere: if A_1B_1 and A_2B_2 are two arcs on the sphere:

$$\vec{n_1} = [\vec{A_1},\vec{B_1}]$$

$$\vec{n_2} = [\vec{A_2},\vec{B_2}]$$

$$\vec{X} = [\vec{n_1},\vec{n_2}]/|\vec{n_1},\vec{n_2}|$$